AIAA JOURNAL
Vol. 46, No. 5, May 2008

Technical Notes

TECHNICAL NOTES are short manuscripts describing new developments or important results of a preliminary nature. These Notes should not exceed 2500 words (where a figure or table counts as 200 words). Following informal review by the Editors, they may be published within a few months of the date of receipt. Style requirements are the same as for regular contributions (see inside back cover).

Stability of Quasi-One-Dimensional Isentropic Flows

Dilip Prasad*
Pratt and Whitney,
United Technologies Corporation,
East Hartford, Connecticut 06108

DOI: 10.2514/1.36486

I. Introduction

THE stability of complex fluid systems can often be estimated to the first order using one-dimensional models, as illustrated by Dowling [1], who describes several model problems related to the prediction of combustion instabilities in gas turbines and rocket engines. A numerical method using a frequency-domain approach similar to that of [1] was recently developed by Prasad and Feng [2] to study the stability of spatially varying one-dimensional flows. In applying this model to several basic flow configurations [3], it was found that isentropic accelerating mean flows tend to be more stable than diffusing ones and, further, that the inlet and exit boundary conditions have an effect on system stability. This interesting behavior appears to have gone essentially unnoticed in the past, and we seek to explore it in more detail here. A semi-analytical asymptotic approach is employed, which enables solutions to be developed in a form that provides insight into the stability behavior. Specifically, the mean diffusion, as measured by a parameter involving the inlet and exit Mach numbers, is found to play a role in this regard. Moreover, the magnitudes of the acoustic reflection coefficients at the inlet and exit boundaries influence the stability of the flow, whereas the phases of these coefficients have a bearing on the natural frequencies of the system eigenmodes. The forms of the expressions for the eigenvalues are shown to lend themselves to a simple physical explanation for the growth rates and frequencies.

II. Model Development and Interpretation

The model employed here is that used by Prasad [4] in a somewhat different context. Specifically, we consider the isentropic flow of an ideal gas with specific heat ratio γ in the region $0 < x < l$, so that governing equations can be written as

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{a}{\beta}\frac{\partial a}{\partial x} = 0 \qquad \frac{\partial a}{\partial t} + u\frac{\partial a}{\partial x} + \beta a\left(\frac{\partial u}{\partial x} + \frac{u}{A}\frac{\mathrm{d}A}{\mathrm{d}x}\right) = 0$$

where $\beta = \frac{1}{2}(\gamma - 1)$, u is the flow velocity, a is the sonic speed, and $A = A(x)$ is the channel cross-sectional area. Defining $\mathbf{Q} = [u \quad a]^T$ and assuming small perturbations of the form $\mathbf{Q}' = \tilde{\mathbf{Q}}e^{i\omega t}$ about a steady mean flow $\bar{\mathbf{Q}}$, we obtain the linearized equations

$$\frac{\partial \tilde{\mathbf{Q}}}{\partial x} + \mathsf{B}^{-1}[\mathrm{i}\omega \mathsf{I} + \mathsf{G}]\tilde{\mathbf{Q}} = 0 \tag{1}$$

where

$$\mathsf{B} = \begin{bmatrix} \bar{u} & \bar{a}/\beta \\ \beta\bar{a} & \bar{u} \end{bmatrix}; \qquad \mathsf{G} = \begin{bmatrix} \frac{\mathrm{d}\bar{u}}{\mathrm{d}x} & \frac{1}{\beta}\frac{\mathrm{d}\bar{a}}{\mathrm{d}x} \\ \frac{\mathrm{d}\bar{a}}{\mathrm{d}x} + \beta\frac{\bar{a}}{A}\frac{\mathrm{d}A}{\mathrm{d}x} & \frac{\beta}{A}\frac{\mathrm{d}}{\mathrm{d}x}(\bar{u}A) \end{bmatrix}$$

and I is the identity matrix. The quantity $\omega = \omega_r + \mathrm{i}\omega_i$ in Eq. (1) is complex, so that the system is stable if $\omega_i > 0$. Next, we assume that the perturbations of interest are such that their length scales are much smaller than those of the mean flow. In particular, defining $\epsilon \ll 1$ to be the ratio of the disturbance and background flow length scales, we introduce the small-scale coordinate $\xi = x/\epsilon$. Because, on the scale of the perturbation, the mean flow is nearly uniform, we may write

$$\omega = \frac{\Omega}{\epsilon} + \mathrm{i}\alpha + \mathcal{O}(\epsilon) \tag{2a}$$

where both Ω and α are real and $\mathcal{O}(1)$. The disturbance field is now described using a multiple-scales perturbation expansion of the form

$$\tilde{\mathbf{Q}} = \tilde{\mathbf{Q}}_0(\xi; x) + \epsilon\tilde{\mathbf{Q}}_1 + \mathcal{O}(\epsilon^2) \tag{2b}$$

Substitution of Eqs. (2a) and (2b) into Eq. (1) yields at $\mathcal{O}(1/\epsilon)$

$$\frac{\partial \tilde{\mathbf{Q}}_0}{\partial \xi} + \mathrm{i}\Omega\mathsf{B}^{-1}\tilde{\mathbf{Q}}_0 = 0 \tag{3}$$

Following the procedure described in [4], the solution to this system is given by

$$\tilde{\mathbf{Q}}_0 = \chi_+(x)\exp\left(-\mathrm{i}\Omega\int\lambda_+\,\mathrm{d}\xi\right)\Psi_+ + \chi_-(x)\exp\left(-\mathrm{i}\Omega\int\lambda_-\,\mathrm{d}\xi\right)\Psi_- \tag{4}$$

where $\lambda_\pm = (\bar{u} \pm \bar{a})^{-1}$ are the eigenvalues of B^{-1}, $\Psi_\pm = [1 \quad \pm\beta]^T$ are the corresponding eigenvectors, and $\chi_\pm(x)$ are functions of x which remain to be determined. The form of the right-hand side of Eq. (4) shows that the perturbation is composed of waves traveling at the local speed of sound along and against the direction of the mean flow. In what follows, these disturbances will be referred to as being of the forward- or backward-propagating type. We note that the expressions on the right-hand side of Eq. (4) become singular if the mean flow velocity approaches the sonic state, $\bar{u} \to \bar{a}$. However, we will restrict ourselves in the present study to subsonic flows in which the Mach number differs sufficiently from unity, so that this issue is not of concern.

Proceeding now to $\mathcal{O}(1)$, we have from Eqs. (1), (2a), and (2b)

$$\frac{\partial \tilde{\mathbf{Q}}_1}{\partial \xi} + \mathrm{i}\Omega\mathsf{B}^{-1}\tilde{\mathbf{Q}}_1 = -\left[\frac{\partial \tilde{\mathbf{Q}}_0}{\partial x} + \mathsf{B}^{-1}\mathsf{G}\tilde{\mathbf{Q}}_0 - \alpha\tilde{\mathbf{Q}}_0\right] \tag{5}$$

The right-hand side of Eq. (5) contains secular terms, which would result in a particular solution of the form $\tilde{\mathbf{Q}}_1 \sim \xi\tilde{\mathbf{Q}}_0$. This implies that $\epsilon\tilde{\mathbf{Q}}_1 = \mathcal{O}(1)$, resulting in a breakdown of the perturbation expansion in Eq. (2b). Suppression of the resonant terms on the right-hand side

Received 5 January 2008; accepted for publication 21 January 2008. Copyright © 2008 by the American Institute of Aeronautics and Astronautics, Inc. All rights reserved. Copies of this paper may be made for personal or internal use, on condition that the copier pay the $10.00 per-copy fee to the Copyright Clearance Center, Inc., 222 Rosewood Drive, Danvers, MA 01923; include the code 0001-1452/08 $10.00 in correspondence with the CCC.

*Aerodynamics Division.

of Eq. (5) is therefore required [4], which yields

$$\frac{d\chi_\pm}{dx} + \lambda_\pm(\Phi_\pm^T \mathsf{G}\Psi_\pm - \alpha)\chi_\pm = 0$$

where $\Phi_\pm^T = \frac{1}{2}[1 \quad \pm\beta^{-1}]$. The solutions to the preceding evolution equations for $\chi_\pm$ can be immediately written as

$$\chi_\pm = \pm\mathcal{A}_\pm \exp\left[\alpha\int\lambda_\pm\,dx - \int(\Phi_\pm^T \mathsf{G}\Psi_\pm)\lambda_\pm\,dx\right] \tag{6}$$

where $\mathcal{A}_\pm$ are the amplitudes of the forward- and backward-traveling waves, which we will assume are specified at $x = 0$ and $x = l$, respectively. The second integral on the right-hand side of Eq. (6) can be determined in closed form by making use of the conservation properties of the background flow, as shown in [4]. Upon making use of these results and reverting to the unscaled variables, we then obtain, correct to $\mathcal{O}(1)$,

$$\tilde{u} = \mathcal{A}_+F_+(x)\exp\left(-i\omega\int_0^x\lambda_+\,dx\right) - \mathcal{A}_-F_-(x)\exp\left(-i\omega\int_l^x\lambda_-\,dx\right) \tag{7a}$$

$$\frac{\tilde{a}}{\beta} = \mathcal{A}_+F_+(x)\exp\left(-i\omega\int_0^x\lambda_+\,dx\right) + \mathcal{A}_-F_-(x)\exp\left(-i\omega\int_l^x\lambda_-\,dx\right) \tag{7b}$$

where the amplitude modulation functions $F_\pm(x)$ are defined according to

$$F_+(x) = \left[\frac{M(x)}{M(0)}\right]^{\frac{1}{2}}\frac{1+M(0)}{1+M(x)}; \qquad F_-(x) = \left[\frac{M(x)}{M(l)}\right]^{\frac{1}{2}}\frac{1-M(l)}{1-M(x)} \tag{8}$$

in which $M = \bar{u}/\bar{a}$ is the Mach number of the mean flow.

The boundary conditions needed to complete the definition of the eigenproblem are general impedance boundary conditions relating the pressure perturbation $\tilde{p}$ to the velocity perturbation $\tilde{u}$. Making use of the relation $\beta\tilde{p} = (\bar{\rho}\,\bar{a})\tilde{a}$, we have

$$\tilde{a}(0) = -\beta Z_0\tilde{u}(0); \qquad \tilde{a}(l) = \beta Z_l\tilde{u}(l) \tag{9}$$

where Z_0 and Z_l represent the inlet and exit acoustic impedances, which are functions of ω_r, in general. Following standard acoustics practice (see, for example, [5]), we will find it useful in the ensuing development to make use of the complex reflection coefficient $\mathcal{R}$, which is related to the impedance by

$$\mathcal{R} = |\mathcal{R}|\exp(i\theta) = \frac{Z-1}{Z+1}$$

When the acoustic resistance is positive [$\mathrm{Re}(Z) > 0$], as is the case for a dissipative element, it follows from the preceding definition that $|\mathcal{R}| < 1$. Next, making use of Eqs. (7–9), we obtain

$$\exp(i\omega\tau) = \mathcal{R}_0\mathcal{R}_l\left[\frac{1+M(0)}{1-M(0)}\right]\left[\frac{1-M(l)}{1+M(l)}\right] \tag{10}$$

where we have defined

$$\tau = \int_0^l(\lambda_+ - \lambda_-)\,dx = \frac{2}{a_0}\int_0^l\frac{1+\beta M^2}{1-M^2}\,dx$$

in which a_0 is the stagnation speed of sound. The imaginary part of Eq. (10) yields

$$\omega_r\tau - [\theta_0(\omega_r) + \theta_l(\omega_r)] = 2n\pi \tag{11}$$

where n takes on integer values. Turning next to the real part of Eq. (10), we obtain

$$\mathrm{Im}(\omega) = -\frac{1}{\tau}\ln\mathcal{D} - \frac{1}{\tau}\ln\{|\mathcal{R}_0(\omega_r)||\mathcal{R}_l(\omega_r)|\} \tag{12a}$$

where

$$\mathcal{D} = \left[\frac{1+M(0)}{1+M(l)}\right]\left[\frac{1-M(l)}{1-M(0)}\right] \tag{12b}$$

is a parameter that quantifies the overall diffusion across the device. Equations (11) and (12) are the central result of the present study, and their implications are examined next.

It is apparent from Eq. (12b) that $\mathcal{D} \lesseqgtr 1$ when $M(0) \lesseqgtr M(l)$. Hence, the first term on the right-hand side of Eq. (12a) is positive in accelerating flows and negative in diffusing ones, showing the tendency of the latter toward instability. However, it is evident from the presence of the second term on the right-hand side of Eq. (12a) that the flow diffusion alone does not determine the stability of the system. In particular, dissipative boundaries, for which $|\mathcal{R}| < 1$, as we have observed previously, tend to provide a stabilizing influence, as one might expect on a physical basis. We also note that when the inlet or exit impedance is unity, Eq. (12a) implies that the damping rate is undefined. In this case, the boundary is nonreflecting and the system cannot support any modes, as was pointed out in [2].

The forms of the expressions in Eqs. (11) and (12a) can be understood in physical terms, as we now illustrate. Consider a disturbance of unit amplitude and frequency f placed just downstream of the inflow boundary at $x = 0+$. The amplitude of the wave generated by this source changes according to Eq. (7) and the first of Eq. (8) as it propagates downstream. The time of flight t_+ of this downstream-propagating wave from $x = 0+$ to the outflow boundary at $x = l$ is

$$t_+ = \int_0^l\frac{dx}{\bar{a}+\bar{u}}$$

so that its phase at the downstream boundary relative to the source is $-2\pi f t_+$. Similarly, it follows from Eq. (8), that the wave amplitude at $x = l$ is

$$\left[\frac{M(l)}{M(0)}\right]^{\frac{1}{2}}\frac{1+M(0)}{1+M(l)}$$

At $x = l$, the disturbance is reflected, undergoing a phase change of θ_l and an amplification of $|\mathcal{R}_l|$. The reflected wave travels upstream, and the time for it to traverse the flow path is

$$t_- = \int_l^0\frac{dx}{\bar{u}-\bar{a}}$$

During this process, the phase of the wave increases by $-2\pi f t_-$, and its amplitude grows by a factor of

$$\left[\frac{M(0)}{M(l)}\right]^{\frac{1}{2}}\frac{1-M(l)}{1-M(0)}$$

At the inflow boundary, the upstream propagating wave is again reflected, causing phase and amplitude changes of θ_0 and $|\mathcal{R}_0|$, respectively. The total time of flight of the disturbance is $t_+ + t_- = \tau$. The overall phase change is therefore $-2\pi f\tau + \theta_0 + \theta_l$. For oscillations to be sustained, the overall phase change must be an integer multiple of 2π, a requirement that leads to Eq. (11). During the total flight time τ, it follows from the preceding arguments that the amplitude increases by a factor of

$$|\mathcal{R}_0||\mathcal{R}_l|\left[\frac{1+M(0)}{1+M(l)}\right]\left[\frac{1-M(l)}{1-M(0)}\right]$$

which is exactly the same result that would be obtained using Eq. (12) for the growth rate. The reason that the disturbance amplitude undergoes an overall change is that its growth or decay during forward propagation is not compensated for by a corresponding attenuation or amplification when it propagates backward, because

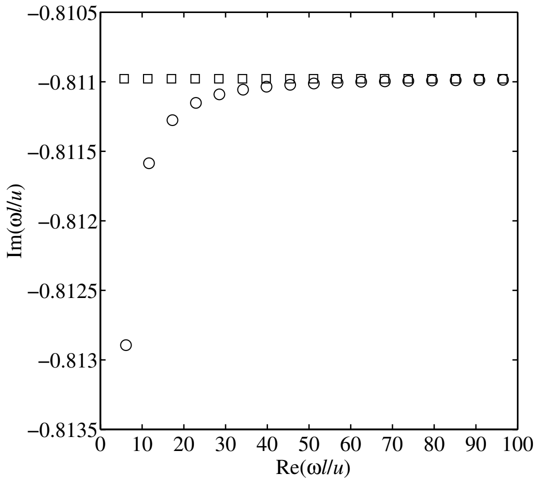

Fig. 1 Comparison of theoretical eigenvalues (squares) with numerical results (circles).

the mean flow performs work on the two types of disturbances differently.

III. Conclusions

In this paper, we have developed an asymptotic model to examine the stability of nonuniform one-dimensional flows in the small-wavelength limit. By taking advantage of the separation of length scales between the mean flow and perturbation, expressions were derived for the system natural frequencies and damping rates. These expressions clearly bring out the influences of the mean diffusion and boundary impedances on the system stability. Moreover, they were shown to lend themselves to a physical explanation for the behavior of the eigenvalues.

It is of interest to compare the asymptotic results with those obtained by numerical solution of Eq. (1). A convenient problem to examine for this purpose is one studied in [3], which considered a diffusing flow with linear variation of M between 0.5 and 0.1, and pressure release conditions at $x = 0, l$. Their eigenvalues, together with the asymptotic estimates, are shown in Fig. 1. The frequencies obtained by the two methods are in very good agreement, although the damping rates exhibit some differences at small mode orders. However, we note that, for this problem, the asymptotic theory still provides a good estimate of the damping rate for the low-order modes.

References

[1] Dowling, A. P., "Calculation of Thermoacoustic Oscillations," *Journal of Sound and Vibration*, Vol. 180, No. 4, 1995, pp. 557–581. doi:10.1006/jsvi.1995.0100

[2] Prasad, D., and Feng, J., "Thermoacoustic Stability of Quasi-One-Dimensional Flows, Part 1: Analytical and Numerical Formulation," *Journal of Turbomachinery*, Vol. 126, No. 4, 2004, pp. 636–643.

[3] Prasad, D., and Feng, J., "Thermoacoustic Stability of Quasi-One-Dimensional Flows, Part 2: Application to Basic Flows," *Journal of Turbomachinery*, Vol. 126, No. 4, 2004, pp. 644–652.

[4] Prasad, D., "Weakly Nonlinear Shock Propagation in Slowly Varying One-Dimensional Flows," *Physics of Fluids*, Vol. 18, No. 3, 2006, p. 036101. doi:10.1063/1.2180288

[5] Munjal, M. L., *Acoustics of Ducts and Mufflers with Application to Exhaust and Ventilation System Design*, Wiley, New York, 1987.

T. Jackson
Associate Editor